López, C. L., Sullivan, H. J. (1992). Effect of personalization of instructional context on the achievement and attitudes of Hispanic students. *Educational Technology Research and Development*, *40*(4), 5–13.

Nesher, P. (1976). Three determinants of difficulty in verbal arithmetic problems. *Educational Studies in Mathematics*, *7*, 369–388.

Nesher, P., Hershkovitz, S., Novotná, J. (2003). Situation model, text base and what else? Factors affecting problem solving. *Educational Studies in Mathematics*, *52*(2), 151–176.

Novotná, J. (2000). *Analýza řešení slovních úloh*. Prague: Charles University, Faculty of Education.

Přibyl, J., Eisenmann, P. (2014). Properties of problem solving strategies. In M. Houška, I. Krejčí, M. Flégl (Eds.), *Proceedings of ERIE 2014* (pp. 623–630). Prague: Czech University of Life Sciences.

Rendl, M., Vondrová, N. et al. (2013). *Kritická místa matematiky na základní škole očima učitelů*. Prague: Charles University, Faculty of Education.

Richard, J.-F. (1984). La construction de la représentation du problème. *Psychologie Française, 29*(21), 226–230.

Searle, B.W., Lorton, P. Jr., Suppes, P. (1974). Structural variables affecting CAI performance on arithmetic word problems of disadvantaged and deaf students. *Educational Studies in Mathematics, 5*, 371–384. doi: 10.1007/BF01424555

Schumacher, R. F., Fuchs, L. S. (2012). Does understanding relational terminology mediate effects of intervention on compare word problems? *J Exp Child Psychol*, *&* (4), 607–628. doi:10.1016/j.jecp.2011.12.001

Verschaffel, L., De Corte, E., Pauwels, A. (1992). Solving compare problems: An eye movement test of Lewis and Mayer's consistency hypothesis. *Journal of Educational Psychology*, *84*, 85–94. doi: 10.1037/0022-0663.84.1.85

Wiest, L. R. (2001). The Role of Fantasy Contexts in Word Problems. *Mathematics Education Research Journal*, *13*(2), 74–90.

Zohar, A., Gershikov, A. (2008). Gender and Performance in Mathematical Tasks: Does the Context Make a Difference? *International Journal of Science and Mathematics Education*, *6*, 677–693.

ON THE REPRESENTATION OF QUANTITIES WITH MULTI-TOUCH AT THE 'MATH-TABLET'

Daniel Walter ☜

Abstract

This paper presents a study, which investigated students' methods of using a tablet-application called 'Math Tablet'. Fifteen students were interviewed, with a focus on those who predominantly solve addition problems through counting strategies. The aim of the investigation was to explore if, and how, students make use of the multi-touch

☞ Technical University of Dortmund, Germany; e-mail: dwalter@math.tu-dortmund.de

potential while representing quantities on the ‘Math Tablet’. Descriptive data analyses show that many learners already use multi-touch to display quantities after short introductory phases. However, it became also clear that some children need targeted impulses, so that they utilise the potential fully. The use of the fingers as a primary hands-on material seems to be more important than the mere use of multi-touch, since the software is to be classified as a secondary hands-on material. Finally, it could also be shown that the child's methods of use can be influenced negatively by some technically evoked difficulties. Accordingly, it becomes clear that the promising potential of the multi-touch technology is not automatically exhausted, but rather appropriate mathematical-oriented accompaniment is necessary.

Keywords: multi-touch, digital media, representation of quantities

Introduction

Since the development of touch-enabled mobile devices started, the discussion about the use of digital media in learning mathematics has been further stimulated. Particularly for the initial arithmetics lessons, the possibility of controlling tablet computers by multi-touch offers new opportunities for representing quantities. However, from a mathematical didactic perspective, little is known about whether and how children access the potentials of digital media in general (e.g. Moyer-Packenham et al., 2015) and the potential of multi-touch technology when using tablet apps in particular (e.g. Ladel and Kortenkamp, 2011). In order to contribute to the closure of this research gap, this article describes empirical findings that provide insights into the use and the thoughts of children when using an app that can be controlled by multi-touch.

This article is structured as follows: In Section 2, the theoretical background to the research work is presented. The focus is both on the description of basic differences between touch-control and traditional use of the mouse and keyboard, as well as the presentation of selected research findings on the use of multi-touch in mathematics teaching and learning. Section 3 contains a description of the research questions and the design of the empirical investigation. Subsequently, the corresponding findings are presented in Section 4. To conclude, the findings are summarized and discussed (Section 5).

Theoretical background

Software can be controlled either traditionally with mouse and keyboard or touch-enabled devices. Intervention studies suggest that when students use software on touch-enabled devices can lead to greater learning outcomes than when using structurally comparable software which is controlled by mouse and keyboard (e.g. Paek, Hoffman and Black, 2013; Segal, 2011). A possible explanation for this result is that learners can perform their actions on touch-enabled devices *directly* on virtual objects. An indirect transmission of the action via the mouse is no longer necessary. The mouse as mediator is absence. While tablet computers require only reliable two-dimensional hand-eye coordination, the operation of a computer is

associated with the ‚difficulty of the triple coordination', since, in addition to eye and hand, the mouse pointer must also be observed on the screen: A place entirely different from the hand (e.g. Ladel, 2016). This can evoke a greater cognitive load than when working with touch apps (Segal, 2011).

In addition to this basic difference between the two operating variants, the central mathematical-didactical difference is the possibility to control tablets with *multiple* fingers. In international literature, multi-touch technology is given great potential to support children in the acquisition of basic mathematical concepts and, in particular, in the representation of quantities (e.g. Baccaglini-Frank and Maracci, 2015; Ladel and Kortenkamp, 2011; Sinclair and Baccaglini-Frank, 2016). Objects can not only be exclusively produced individually, as it is often the case when working with traditional software, which is operated with a mouse and keyboard. Multi-touch technology also allows students to add multiple objects simultaneously using multiple fingers on the touch screen. Children are thus given the opportunity to represent quantities not only sequentially, but also in the sense of the part-whole concept (Resnick, 1983).

Various research works provide information on whether and how children use this potential and which difficulties can arise. In their experiments on the use of the multi-touch-table (MTT), Ladel and Kortenkamp (2011, 2012, 2014) investigated how internalization and externalization processes can proceed on this digital medium. It has been found that the formulation of a task can have an impact on the use of the child. For example, children tended to assign *individual* tokens on tasks such as "Please put x tokens on the table". However, if the children were additionally encouraged to display the tokens "all at once," many learners changed their approach by representing quantities quasi-simultaneously with fingers. In addition, the authors refer to different methods of use:

> „Some children first counted their fingers one by one and then put them all at once on the table. Other children did the opposite, showing fingers all at once when asked for a certain number, and working one-by-one on the MTT" (Ladel and Kortenkamp, 2014, p. 250).

Accordingly, it must always be taken into account that *both* fingers and digital media (here: MTT) are representative media. The quasi-simultaneous representation of quantities at the MTT does not necessarily indicate a cardinal number concept when a finger set was previously derived sequentially. At the same time, the sequential representation of quantities on the MTT can not be automatically characterized as a sequential conception if the finger set was previously determined quasi-simultaneously. Thus, the existence of multi-touch in software does not guarantee its adequate use by children. It is also possible to display objects one at a time via single-touch, which is structurally consistent with successive single mouse clicks (e.g. Ladel and Kortenkamp, 2014). Previous research has also shown that children can also have difficulties with handling touch-enabled software. In the study of Sinclair and Heyd-Metzuyanim (2014),

some children touched the screen surface while representing quantities on the app *TouchCounts* unintentionally with more fingers than wanted. So more counters appeared on the screen as required. This difficulty can particularly be seen as an obstacle if the child can not locate faulty touch inputs itself and considers incompatible representation as belonging to one another. The development of erroneous numerical representations can be a consequence of this. Some difficulties with the MTT could be spotted in the studies of Ladel and Kortenkamp as well. In some cases, the children did not exercise enough pressure with their fingers to produce tokens, so they often switched to the one-by-one method to sequentially display tokens with the index finger (Ladel and Kortenkamp, 2014). Furthermore, in the experiments by Barendregt and colleagues (2012), some children had motoric difficulties while using the app *Fingu* for representing quantities, too (Barendregt et al., 2012).

Overall, it is important to note that the possibilities of multi-touch and the chance of *Direct Manipulation* (e.g. Sarama and Clements, 2016) of objects with regard to virtual-enactive representations are becoming increasingly important (e.g. Krauthausen, 2012). This is especially evident in the ever-growing range of tablet apps that allow multi-touch operations. On the theoretical level, multi-touch seems to be a promising design element that can support the advancement of children in mathematics. However, the consequences and implications of this technical innovation for mathematics teaching and learning have not yet been sufficiently explored. In addition, possible further difficulties and hurdles in the handling of other soft- and hardware must be identified in order to elaborate appropriate strategies for dealing with them.

Research questions and the design of the investigation

Research questions

In Section 2, theoretical considerations and empirical findings on the usage of the multi-touch potential in mathematics teaching and learning were presented. Beyond the described research findings, little is known about methods of use while dealing with multi-touch software – especially tablet-apps – for representing quantities. Appropriately, the following two research questions form the starting point for the empirical investigation:

- To what extent is the potential of multi-touch technology used by students for representing quantities?
- What are the particularities in children's representation processes when using multi-touch-capable software?

The tablet-app ‘Math Tablet’

As an example for multi-touch capable tablet apps, which have not yet been fully evaluated, the tablet app 'Math Tablet' (Urff, n. y.) was used in the study described.

Counters can be displayed in two fields, separated by a line, by touch control. There are as many counters as fingers touching the screen (see Figure 1). The corresponding numeric symbols for each field are displayed in accordance with the iconic representation. In addition, the number of all placed counters is also represented in numerical form.

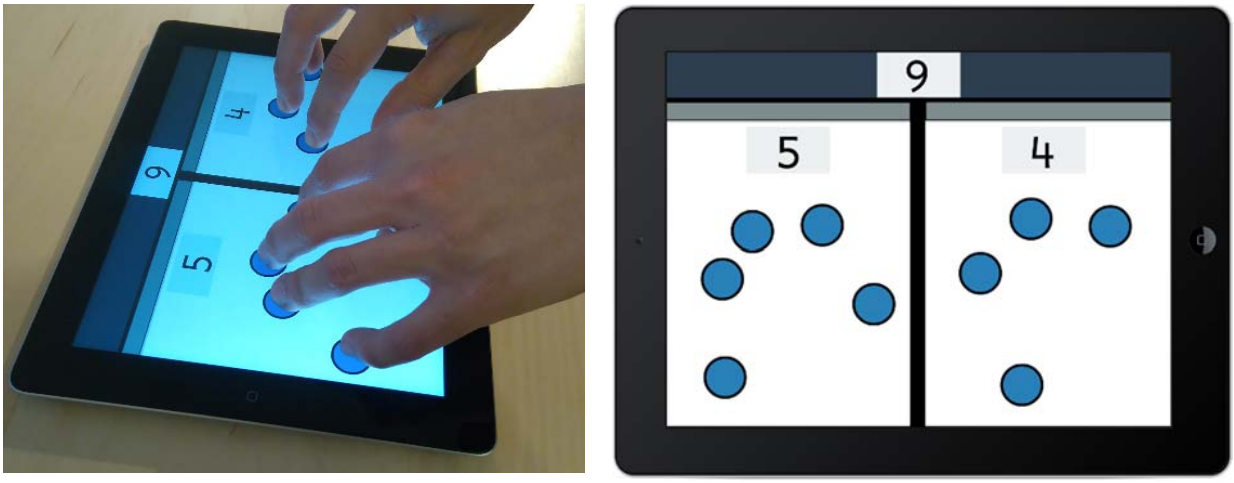


Figure 1: Multi-touch interactions at the ‘Math Tablet’

Process of the investigation and interview tasks

In order to investigate whether and how children use the multi-touch technology at the 'Math Tablet' in the course of the representation of quantities, this process was investigated by means of exemplarily selected quantities in a qualitative interview. In the present study, quantities consisting of 9 and 15 elements were used. For both, it seems to be expedient to add multiple counters simultaneously. For the representation of quantities the children were at first asked to present counters only in the right one of the two fields. This purpose of this approach is to investigate whether the dimensions of a field with standard hardware (9.7 x 12 cm for a 4th generation iPad) are sufficiently large to use the multi-touch technology adequately. In the event that a child represented a number in the first attempt purely sequentially with only individual counters, the child was asked to display the corresponding number of counters again, but this time in the left field "in as few steps as possible". It was also emphasized that they can add multiple counters, too. In this way the children were explicitly encouraged to use the multi-touch technology. At the beginning of the interview, a ten- to fifteen-minute introduction to working with the app took place, which was not in the form of an instruction, but a joint preparation. In this phase, the children have discovered that multiple counters *can* be added simultaneously. It has also been discussed, among other things, why it can be helpful to use multi-touch. The interviews were videotaped from two perspectives. While one camera provides a front view of the child and the interviewer, the second camera is positioned next to the child and

directed at the 'Math Tablet'. Thus, both the mimic and gesture as well as the actions on the tablet could be observed.

Information on the sample and data assessment

A total of 15 children ages 7 to 8 took part in the interview described. The children have learning difficulties in mathematics, at the beginning of their second school year and solved simple addition tasks preferably with counting strategies. None of the examined children knew the 'Math Tablet' before the interview, whereby the fluency of the functions and usage of the app could and should be trained only by the introductory phase. Accordingly, the observed methods of use can be regarded as the first intuitive approaches of the children.

The assessment methods used were qualitative content analysis (Mayring, 2015) and comparative analysis (Glaser and Strauss, 2005). Based on the data material, categories were developed for the different methods of students' use of the application 'Math Tablet'. Accordingly, inductive category formation was undertaken which developed into a structured analysis of the contents as the assessment process continued.

Results

This section presents selected results of the investigation along the described research questions.[1] Section 4.1 deals with the question of whether the children used the multi-touch technology when representing quantities. Subsequently, in Section 4.2 the particularities while using multi-touch are described.

Representation of quantities on the 'Math Tablet'

Table 1 shows the absolute number of students who used multi-touch or only single-touch to represent quantities consisting of nine or fifteen elements. The data show that a total of nine children used only individual counters to represent a set consisting of nine elements. Six children used the multi-touch potential and added multiple counters simultaneously. In representing a set consisting of fifteen elements, there were six children who initially added only individual counters. Conversely, nine children used the multi-touch technology. Those children who only added individual counters in the first attempt were encouraged to add several counters simultaneously, as described in Section 3, in order to represent the respective amount in as few steps as possible. For this second attempt, the data show that all children used the multi-touch technology after explicit proposal in the process of representation. In light of the empirical findings, some children have already used multi-touch in their first intuitive approaches to represent quantities. According to explicit proposal, all children used the multi-touch study described at least once. This finding underlines the special importance of suitable tasks and impulses.

[1] The investigation results shown in this Section are taken from Walter (2017/ in preparation).

Number of students, who ...	9 counters		15 counters	
	First attempt	**Second attempt**	**First attempt**	**Second attempt**
... only added single counters .	9	0	6	0
... added counters by means of multi-touch.	6	9	9	6

Table 1: Usage of multi-touch technology for representing numbers

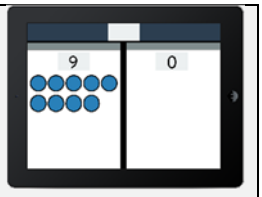
Detail analysis of the representation processes

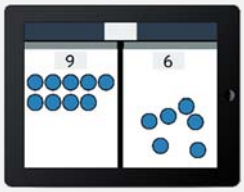
In order to learn more about the processes and not only about the question of whether multi-touch was used, three specific particularities are described below which have often been observed in children's methods of use.

1. *Significance of the fingers as a primary representation medium*

The quantity representation on the 'Math Tablet' is done as described by creating counters with the input of fingers. Accordingly, the children work with two different media: their fingers and the 'Math Tablet'. A first particularity in the representation of quantities was that the simultaneous addition of multiple counters on the 'Math Tablet', which corresponds to a cardinal representation of quantities, is no guarantee that the representation of quantities with fingers was previously cardinal, too. As the following part of an interview shows, children can first display a quantity with their fingers sequentially and then try to touch the screen with all their fingers at the same time. The scene starts when the child has shown a set of nine elements in the left field only sequentially and then asked to represent the quantity in as few steps as possible in the right field.

1	I	Try again!
2	M	(*stretches nine fingers one by one and counts quietly*) One, two, three, four, five, six, seven, eight, nine.
3	M	(*touches her chin with the nine outstretched fingers one by one and counts quietly again*) One, two, three, four, five, six, seven, eight, nine.
4	M	(*moves the nine fingers simultaneously to the screen to place nine counters. Only six fingers touch the screen.*)





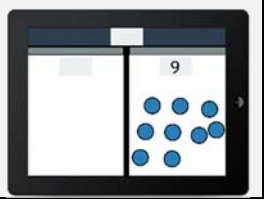
None of the children investigated in this study represented a quantity on the fingers sequentially and then touched the screen with all fingers. Accordingly, it is essential to emphasize the importance of the fingers as a *primary* medium of representation, as the visual representation processes on this medium tend to

reflect children's thinking rather than the use of the 'Math Tablet'. This thus represents the *secondary* medium of representation.

2. *(Subjectively) more reliable quantity representation by using single-touch*

As explained, some children did not use multi-touch consistently and represented counters only individually with single-touch. A possible explanation for this may be that at least some children felt subjectively safer in the addition of individual counters. The following transcript describes a student's approach during the presentation of a quantity that consists of nine elements.

1	I	Can you (.) put nine counters in the right field? (*points to the right part of the ‚Math Tablet'*)
2	Z	(*adds one single counter)* Hmm (*deletes the counter again*)
3	Z	(*stretches out all five fingers of the right hand and four fingers of the left hand simultaneously. Then she folds the four fingers of the left hand one after the other and then stretches three fingers one after the other. Thus, a total of eight fingers are stretched*.)
4	Z	(*moves all eight outstretched fingers towards the right field. Shortly before her fingers touch the screen, she stops and only extends the index finger of the right hand*.)
5	Z	(*places nine counters one by one with the right index finger in the right field*)



After the task was given Zoe begins the representation process. Because of a count error, she does not stretch out nine, but only eight fingers. She moves them to the tablet to touch the screen with all her fingers simultaneously. But just before the fingers touch the tablet, she deviates from the intended procedure and represents all counters sequentially with the index finger. Zoe rejected her planned approach with multi-touch in favor of sequential representation with single-touch. One possible reason for this is that she did not have sufficient space in a single field. Possibly, she herself stated that she could not provide the desired result with the position of her fingers (see Figure 2), whereupon she switched to the sequential addition of individual counters, which in her opinion was a more accessible and safer variant of the representation of the required quantity. Similar

Figure 2: Zoe's finger positioning (see turn 4)

scenes were also observed during interviews with other children. Some of them explicitly pointed to the limited place of the app.

3. *Difficulties in representing of quantities with multi-touch*

In the transcribed scenes so far, it became clear that some children had difficulties in representing multiple counters by means of multi-touch. It often occured that children represented more or less counters than originally intended. More counters were created by the children touching the screen with their stretched fingers as well as with other fingers or the heel of hand inadvertently. Less counters than wanted were often represented because the children held two fingers too close together. In this case, the software generated only one counter for two touch inputs. Furthermore, several children did not manage to position their hands in such a way that only the fingertips touch the screen after repeated attempts to add several counters. As a result, students often produced two incompatible representations. Some children recognized their mistakes and fixed them immediately. However, other children saw the incompatible representations as belonging to one another. Even though they were initially sceptical. They often tended to rely on the representations produced by the "Math Tablet" rather than on their own mathematical concepts.

Closing remarks

The multi-touch technology opens up new mathematical-didactical possibilities, which lead to promising opportunities for the promotion of basic mathematical competences. The empirical findings described show, on the one hand, that all children, after a brief introduction, integrated this potential in their methods of use. On the other hand, it became clear that some children needed specific stimulus to use the multi-touch potential. To display nine counters nine children used single-touch, while six children did so for fifteen counters. However, in the case of the use of multi-touch, it must also be considered that the potential is not automatically exhausted. It could often be observed that quantities were displayed by stretching fingers sequentially and then added via multi-touch. Likewise, difficulties with multi-touch were identified that led children to use less appropriate approaches. A proper handling of the fingers as a mathematical hands-on material as well as the awareness of possible technically evoked difficulties is necessary. Under these conditions, there seem to be opportunities to enrich mathematics teaching and learning with this potential.

References

Baccaglini-Frank, A., Maracci, M. (2015). Multi-Touch Technology and Preeschooler's Development of Number-Sense. *Digital Experiences in Mathematics Education, 1*, 7–27.

Barendregt, W., Lindström, B., Rietz-Leppänen, E., Holgersson, I., Ottosson, T. (2012). Development and Evaluation of Fingu: A Mathematics iPad Game Using Multi-touch Interaction. *IDC 2012, Germany*, 1–4.

Glaser, B. G., Strauss, A. L. (2005). *Grounded theory. Strategien qualitativer Forschung* (Vol. 2). Bern: Verlag Hans Huber.

Krauthausen, G. (2012). *Digitale Medien im Mathematikunterricht der Grundschule* [Digital media in primary school mathematics]. Heidelberg: Springer Spektrum.

Ladel, S. (2016). Lehren und Lernen von Mathematik mit digitalen Medien – ein Blick in die (nahe) Zukunft. In M. Peschel, T. Irion (Eds.), *Neue Medien in der Grundschule 2.0: Grundlagen – Konzeption – Perspektiven* (pp. 247–258). Frankfurt am Main: Grundschulverband.

Ladel, S., Kortenkamp, U. (2011). Finger-symbol-sets and multi-touch for a better understanding of numbers and operations. In M. Pytlak, T. Rowland, E. Swoboda (Eds.), *Proceedings of CERME 7* (pp. 1792–1801). Rzeszów: University of Rzeszów.

Ladel, S., Kortenkamp, U. (2014). Number concepts – processes of internalization and externalization by the use of multi-touch technology. In C. Benz, B. Brandt, U. Kortenkamp, G. Krummheuer, S. Ladel, R. Vogel (Eds.), *Early Mathematics Learning. Selected Papers of the POEM 2012 Conference* (pp. 237–256). New York: Springer.

Mayring, P. (2015). *Qualitative Inhaltsanalyse* [Qualitative content analysis] (Vol. 12). Weinheim und Basel: Beltz.

Moyer-Packenham, P. S., Shumway, J. F., Bullock, E., Tucker, S. I., Anderson-Pence, K. L., Westenskow, A., Jordan, K. (2015). Young Children's Learning Performance and Efficiency when Using Virtual Manipulative Mathematics iPad Apps. *Journal of Computers in Mathematics and Science Teaching, 34*(1), 41–69.

Paek, S., Hoffman, D. L., Black, J. B. (2013). *Using Touch-Based Input to Promote Student Math Learning in a Multimedia Learning Environment*. Paper presented at the EdMedia 2013.

Resnick, L. B. (1983). A Developmental Theory of Number Understanding. In H. P. Ginsburg (Ed.), *The Development of Mathematical Thinking* (pp. 109–151). New York: Academic Press.

Sarama, J., Clements, D. H. (2016). Physical and Virtual Manipulatives: What is "Concrete"? In P. S. Moyer-Packenham (Ed.), *International Perspectives on Teaching and Learning with Virtual Manipulatives* (pp. 71–94). Switzerland: Springer.

Segal, A. (2011). *Do gestural interfaces promote thinking? Embodied interaction: Congruent gestures and direct touch promote performance in math*: ProQuest Dissertations Publishing.

Sinclair, N., Baccaglini-Frank, A. (2016). Digital Technologies in the Early Primary School Classroom. In L. D. English, D. Kirshner (Eds.), *Handbook of International Research in Mathematics Education* (pp. 662–686). New York: Taylor & Francis.

Sinclair, N., Heyd-Metzuyanim, E. (2014). Developing number sense with TouchCounts. In S. Ladel, C. Schreiber (Eds.), *Von Audiopodcast bis Zahlensinn* (pp. 125–150). Münster: WTM-Verlag.

Urff, C. (n. y.). Digitale Lernmedien für die Grundschulstufe [Virtual working equipment in primary stage]. Retrieved from www.lernsoftware-mathematik.de.

Walter, D. (2017/ in preparation). *Nutzungsweisen bei der Verwendung von Tablet-Apps* [Methods of use when using apps]. Dissertation: TU Dortmund.